# Functions

- *def:* A function $f$ from set $\mathbf{X}$ to set $\mathbf{Y}$ is a relationship between the elements of the sets $\mathbf{X}$ and $\mathbf{Y}$ in which each element of $\mathbf{X}$ is related to a unique element of $\mathbf{Y}$. It is denoted by $f : \mathbf{X} \to \mathbf{Y}$.

  *Function application:* We use the notation $f(x)$, which is read "*$f$ of $x$*", to represent the element $y$ of $\mathbf{Y}$ which is related to $x$ by the function $f$. The variable $x$ is called the argument of the function $f$.

- *def:* Given an element $a$ in $\mathbf{X}$, there is a unique element $b$ in $\mathbf{Y}$ that is related to $a$ and which is denoted by $f(a)$. We call this $b$ the *image of* $a$ under $f$, and $a$ is called the preimage of $b$.

- *def:* For a function $f$ from $\mathbf{X}$ to $\mathbf{Y}$ the set $\mathbf{X}$ is called the *domain* of $f$ and $\mathbf{Y}$ is called the *co-domain* of $f$. The *range* of the function is the set of all images of elements in $\mathbf{X}$.

## Multi-variable functions

- *def:* Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be sets.

  A function $f$ from $\mathbf{X} \times \mathbf{Y}$ (the Cartesian product of the sets $\mathbf{X}$ and $\mathbf{Y}$) is a relationship between the elements of $\mathbf{X} \times \mathbf{Y}$ and the set $\mathbf{Z}$ where each element of $\mathbf{X} \times \mathbf{Y}$ is related to a unique element of $\mathbf{Z}$. It is denoted by $f: \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$.

- *def:* Let $\mathbf{X}_1$, $\mathbf{X}_2$, ..., $\mathbf{X}_n$ and $\mathbf{Y}$ be sets.

  A function $f$ from $\mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n$ is a relationship between the elements of $\mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n$ and the set $\mathbf{Y}$ where each element of $\mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n$ is related to a unique element of $\mathbf{Y}$. It is denoted by $f: \mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n \to \mathbf{Y}$.

  The application of $f$ to the arguments $x_1, x_2, \ldots, x_n$ is represented by the notation

$$f(x_1, x_2, \ldots, x_n)$$

## Defining functions (1)

1. **Arrow diagram:** If **X** and **Y** are finite sets, an *arrow diagram* depicts a function $f$ from **X** to **Y** by drawing an arrow from each element in **X** to the corresponding element of **Y**.
   - every element $x$ of **X** must have an arrow coming out of it.
   - No element of **X** can have two or more arrows coming out of it.

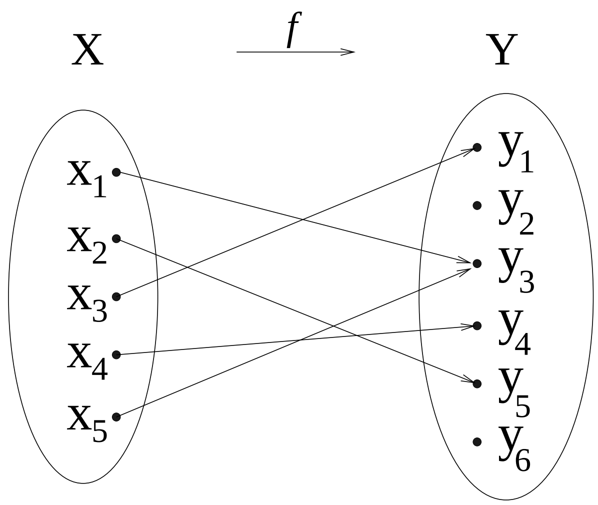


Figure 1: A function shown by an arrow diagram

# Defining functions (2)

**2. Analytical form:** A more general way of defining a function is to give a formula defining the result of the function $f(x)$ in terms of the argument $x$. This formula is called the *analytical form* of the function.

- **Identity function on a set**
  The *identity function* on a set $\mathbf{X}$

  $$i_x : \mathbf{X} \to \mathbf{X}$$

  is defined by $i_x(x) = x$.

- **Absolute value function**
  is the function $f : \mathbf{R} \to \mathbf{R}$ (denoted by $|x|$) such that

  $$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- **Polynomial function of degree $n$**
  is a function $f : \mathbf{R} \to \mathbf{R}$ such that

  $$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

# Defining functions (3)

**3. Function machines:** Functions can be thought of as machines or computer programs. Suppose $f$ is a function from $\mathbf{X}$ to $\mathbf{Y}$ and an input $x$ of $\mathbf{X}$ is given. Then $f$ can be imagined as a machine (or program) that processes $x$ in a certain way to produce the output $f(x)$.

**Example:** Let $\mathbf{S} = \{0, 1\}$ and $n \in \mathbf{N}$ and let $\mathbf{S}^n$ be the set of all string of 0's and 1's of length $n$. Then the *Hamming distance function* $h : \mathbf{S}^n \times \mathbf{S}^n \to \mathbf{N}$ returns, for a pair in $\mathbf{S}^n \times \mathbf{S}^n$, the number of positions at which the two strings of the pair have different values, i.e.

$$h(s,t) = \text{the number of positions at which } s \text{ and } t \text{ have different values.}$$

For example,

$$h(110001, 110010) = 2.$$

A computer program (or procedure) which evaluates the distance between two such strings is a function.

# One-to-one functions (1)

- *def:* The function $f: \mathbf{X} \to \mathbf{Y}$ is called a *one-to-one function* (or *injection*) if, and only if, any two distinct elements $x_1$ and $x_2$ of $\mathbf{X}$ have distinct images under $f$: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- In terms of arrow diagrams, a one-to-one function takes distinct points of the domain to distinct points of the co-domain.

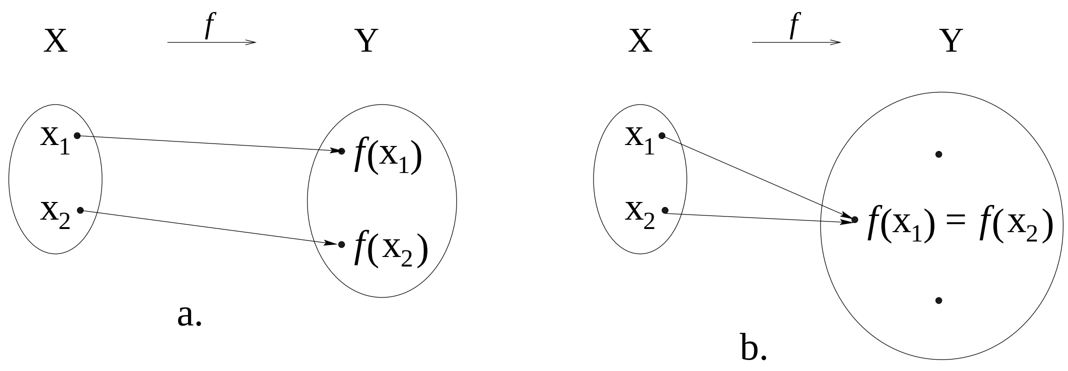


Figure 2: a. A one-to-one function. b. A function that is not one-to-one

## One-to-one functions (2)

- **One-to-one functions on infinite sets:** To prove a function is one-to-one the method of direct proof is generally used.

  **Example:** Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by the rule $f(x) = 6x - 2$

  Prove that $f$ is one-to-one on $\mathbf{R}$.

  *Proof:* Suppose $f(x_1) = f(x_2)$. We need to show $x_1 = x_2$.

$$\begin{aligned} 6x_1 - 2 &= 6x_2 - 2 \\ 6x_1 &= 6x_2 \\ x_1 &= x_2 \end{aligned}$$

  as desired.

# Onto functions

- *def:* Let $f: \mathbf{X} \to \mathbf{Y}$.
  The function $f$ is said to be *onto* (or *surjective*) if, and only if, each element in $\mathbf{Y}$ is the image of some element of $\mathbf{X}$ under the function $f$:

  $f$ is onto if, and only if,
  $\forall y(y \in \mathbf{Y} \to \exists x(x \in \mathbf{X} \land f(x) = y))$ is true.

  **Example:** Let $f: \mathbf{R} \to \mathbf{R}$ be a function defined by the rule

  $$f(x) = 6x - 2$$

  Prove that $f$ is onto on $\mathbf{R}$.

  *Proof:* Let $y \in \mathbf{R}$. We need to show that there exists $x \in \mathbf{R}$ such that $f(x) = y$.
  If such a real number exists, then

  $$6x - 2 = y$$
  $$\text{Hence, } x = (y + 2)/6.$$

  $x$ is a real number. It follows that $f$ is onto.

## One-to-one correspondences

- *def:* A function is called a *one-to-one correspondence* or *bijection* if it is both *one-to-one* and *onto*.

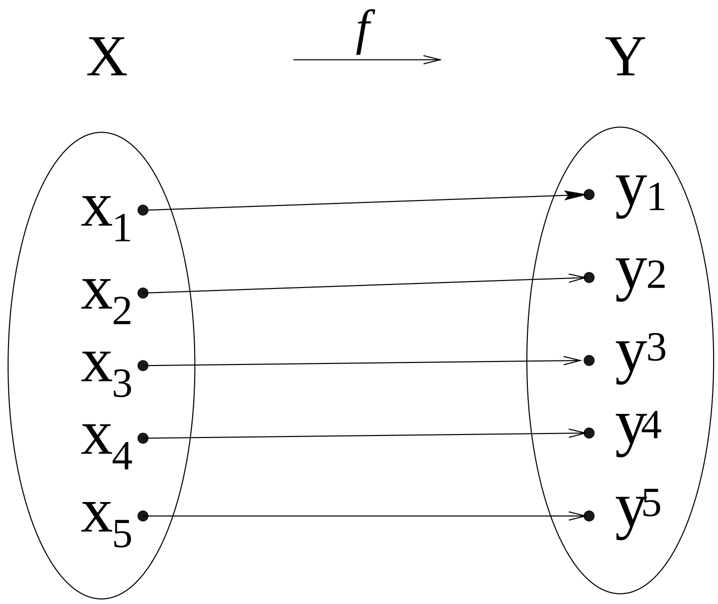


Figure 3: A function which is a one-to-one correspondence

# Composition of functions

Let $g : \mathbf{X} \to \mathbf{Y}_1$ and $f : \mathbf{Y} \to \mathbf{Z}$ be functions such that $\mathbf{Y}_1 \subseteq \mathbf{Y}$.

• *def:* The *composition* of functions $f$ and $g$, which is denoted $f \circ g$, is a function from $\mathbf{X}$ to $\mathbf{Z}$ such that

$$(f \circ g)(x) = f(g(x)) \text{ for all } x \text{ in } \mathbf{X}.$$

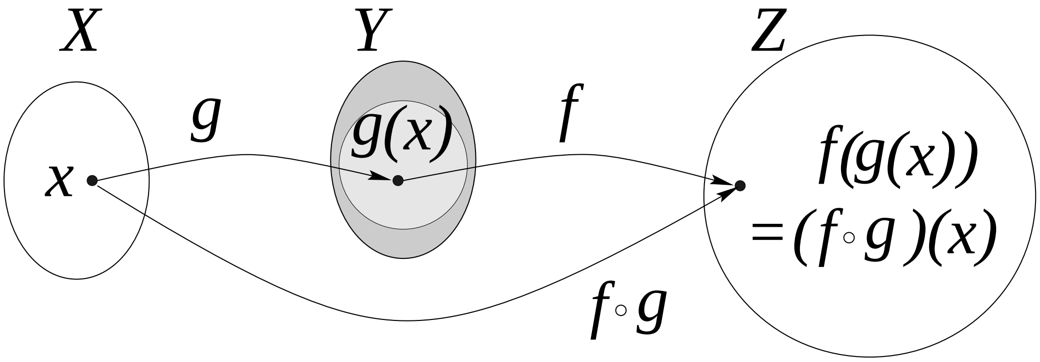


Figure 4: Composition of two functions

**Example:** Let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = x^2 + 2$, and let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x + 4$. Then

$$(f \circ g)(x) = f(g(x)) = 3(x^2 + 2) + 4 = 3x^2 + 6 + 4 = 3x^2 + 10$$

$$(g \circ f)(x) = g(f(x)) = (3x + 4)^2 + 2 = 9x^2 + 24x + 16 + 2 = 9x^2 + 24x + 18.$$