## Inductive definitions (1)

- *def:* Let $\mathbf{S}$ be an infinite set to be defined, and let $n$ and $m$ be finite natural numbers. An *inductive definition* of $\mathbf{S}$ is a definition with the following three properties:
  1. **Base cases:** Objects $b_1, b_2, \ldots, b_n$ are elements of the set $\mathbf{S}$, i.e. $b_i \in \mathbf{S}$ for all $i$ such that $1 \leq i \leq n$
  2. **Constructor functions:** The functions
     $g_1 : \mathbf{S} \rightarrow \mathbf{S}$
     $g_2 : \mathbf{S} \rightarrow \mathbf{S}$
     $\vdots$
     $g_m : \mathbf{S} \rightarrow \mathbf{S}$
     establish ways to obtain new elements from some of the previously defined elements of the set $\mathbf{S}$.
  3. **Induction axiom** asserts that nothing else is in the set $\mathbf{S}$ other than elements obtained by applying (1) and (2).

$$p(b_1) \wedge p(b_2) \wedge \cdots \wedge p(b_n) \wedge$$
$$\forall e \in \mathbf{S} \bullet (p(e) \Rightarrow p(g_1(e)) \wedge p(g_2(e)) \wedge \cdots \wedge p(g_n(e))) \Rightarrow$$
$$(\forall e \in \mathbf{S} p(e))$$

## Inductive definitions (2)

- Suppose that the universe is the set $\mathbf{N}$. Then the set $\mathbf{E}$ of even integers can be defined inductively as follows:

  1. Base case: 0 is in $\mathbf{E}$, i.e. $0 \in \mathbf{E}$.
  2. Constructor functions: The functions $g_1(n) = n - 2$ and $g_2(n) = n + 2$ both generate even integers. That is, if $n$ is in $\mathbf{E}$ then $g_1(n) \in \mathbf{E}$ and $g_2(n) \in \mathbf{E}$.
  3. Extremal clause:
     $\forall e \in \mathbf{E}\ (e = 0 \vee \exists n \in \mathbf{E}(e = g_1(n) \vee e = g_2(n)))$

- Consider the Fibonacci sequence of numbers:

  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

  1. Base cases: $F(0) = 1$ and $F(1) = 1$ are both members of the sequence.
  2. Constructor function: $F(n) = F(n-1) + F(n-2)$ for all $n \geq 2$

## First principle of mathematical induction

Let $p(n)$ be a predicate on $\mathbf{N}$.

- *def:* The *first principle of mathematical induction* is an inference rule of the form:

  **If** $p(0) \wedge \forall k\,(p(k) \rightarrow p(k+1))$,
  **then** $\forall n\, p(n)$

- **Proof technique:**

  1. Basis: Show that $p(0)$ is true.
  2. Induction: Let $k$ be an arbitrary element of $\mathbf{N}$. Assume that $p(k)$ is true (this is the induction hypothesis) and show that $p(k+1)$ is true.

**Example:** Prove that $S(n) \equiv \sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Let $p(n)$ be the predicate stating that the formula is true for $n$.

## Proof by mathematical induction (1)

1. Basis: Since $S(0) = 0^3$ and $(0(0+1)/2)^2 = 0$, $p(0)$ is true.
2. Induction: Assume that $p(k)$ is true, that is
   $s(k) \equiv \sum_{i=0}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$.
   Now we show that $p(k+1)$ is true.

   Observe that $S(k+1) = S(k) + (k+1)^3$.

   $$\begin{aligned} S(k+1) &= S(k) + (k+1)^3 \\ &= (k(k+1)/2)^2 + (k+1)^3 \\ &= (k+1)^2(\,(k/2)^2 + (k+1)) \\ &= (k+1)^2(k^2/4 + k + 1) \\ &= (k+1)^2(k^2 + 4k + 4)/4 \\ &= (k+1)^2(k+2)^2/4 \\ &= ((k+1)(k+2)/2)^2 \end{aligned}$$

   We arrive at $S(k+1) = ((k+1)(k+2)/2)^2$ which shows that $p(k+1)$ is true.

## Second principle of mathematical induction

Let $p(n)$ be a predicate on $\mathbf{N}$.

- *def:* The *second principle of mathematical induction* is an inference rule of the form:

  **If** $\forall n\, (\forall k\, (k < n \wedge p(k)) \rightarrow p(n))$,
  **then** $\forall n\, p(n)$

- **Proof technique:** Let $n$ be an arbitrary element in $\mathbf{N}$.
  Assume that $p(k)$ is true for every $k < n$ and show that $p(n)$ is true.

## Proof by mathematical induction (2)

**Example:** Any integer $n \geq 2$ can be written as a product of primes.

Let $p(n)$ be the predicate 'n can be written as a product of primes'.
Assume that $p(k)$ is true for every $2 \leq k < n$. Then we show that $p(n)$ is true.

The proof is by cases:
*Case 1:* $n$ is a prime.
Then $n$ is a product of one prime, namely itself, i.e. $p(n)$ is true.

*Case 2:* $n$ is not a prime.
Then $n$ must have a factor $i$ which satisfies $2 \leq i < n$.
So we can write $n = i \cdot j$ where $j$ also satisfies $2 \leq j < n$
By the induction hypothesis, both $i$ and $j$ can be written as products of primes.
Therefore, $n$ can be written as a product of the products of these primes, i.e. $p(n)$ is true.

In both cases, $p(n)$ is true, so $p(n)$ is true for all $n \geq 2$.