Mappings

Let $\mathbf{A}$ and $\mathbf{B}$ be sets.

- *def:* A *mapping* from set $\mathbf{A}$ to set $\mathbf{B}$ is a relationship between all elements of $\mathbf{A}$ and some or all elements of $\mathbf{B}$ in which each element of $\mathbf{A}$ is related to a unique element of $\mathbf{B}$.

- *def:* For a mapping from $\mathbf{A}$ to $\mathbf{B}$, if s is an element of $\mathbf{A}$ (i.e. $s \in \mathbf{A}$) and r is the element of $\mathbf{B}$ that is related to s in the mapping, then r is called the *image* of s under the mapping, and we say that the mapping *maps* s to r and denote this by $s \mapsto r$.

- *def:* Let m be a mapping from $\mathbf{A}$ to $\mathbf{B}$ and let $a \in \mathbf{A}$. Then the image of the element a under m is denoted by $m(a)$ and the following holds: $\forall a \, (a \in \mathbf{A} \rightarrow m(a) \in \mathbf{B})$

- *def:* For a mapping from $\mathbf{A}$ to $\mathbf{B}$ the set $\mathbf{A}$ is called the *domain* and the set $\mathbf{B}$ the *range* of the mapping.

- *def:* We denote the domain of a mapping m by $dom(m)$ and its range by $rng(m)$.

Defining mappings (1)

1. **Enumeration:** We write a mapping by listing its associations between brackets [ ] explicitly.
   - $m_1 = [A \mapsto 65, B \mapsto 66, C \mapsto 67, a \mapsto 97, b \mapsto 98, c \mapsto 99]$

     $m_1 = [A \mapsto 65, a \mapsto 97, B \mapsto 66, b \mapsto 98, C \mapsto 67, c \mapsto 99]$
   - For the mapping
     $m_2 = [2 \mapsto true, 3 \mapsto false, 4 \mapsto true, 5 \mapsto false]$
     its domain and range are $dom(m_2) = \{2, 3, 4, 5\}$ and $rng(m_2) = \{true, false\}$
2. **Map comprehension** We can define a mapping using a notation similar to the one for set comprehension based on the fact that a mapping is a set of associations:

   $$[f(s) \mapsto g(s) \mid s \in \mathbf{S} \wedge p(s)]$$

   where f and g are some functions of s. This constructs the set of associations of the form $f(s) \mapsto g(s)$ for all s which satisfy the given predicate p.

Defining mappings (2)

Examples

$$[n \mapsto -n \mid n \in \mathbf{Z} \wedge (-1 \leq n \leq 1)] = [-1 \mapsto 1, 0 \mapsto 0, 1 \mapsto -1]$$

$$[n \mapsto 2 \cdot n - 1 \mid n \in \mathbf{N} \wedge (n \leq 4)] =$$
$$[0 \mapsto -1, 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 5, 4 \mapsto 7]$$

- *def:* The *empty* mapping is the mapping whose domain is the empty set and which thus contains no associations. It is represented by $[\,]$.

  The mapping
  $[n \mapsto 2 \cdot n - 1 \mid n \in \mathbf{N} \wedge (n < 0)]$
  is the empty mapping because its domain is the set
  $\{n \in \mathbf{N} \mid n < 0\}$
  which is empty.

Properties of mappings

- *def:* A mapping is *injective* if for each element r of the range of the mapping there is at most one element of the domain whose image under the mapping is r.
- *def:* A mapping is *surjective* if for each element r of the range of the mapping there is at least one element of the domain which maps to r under the mapping.
- *def:* A mapping is a *bijection* if it is both injective and surjective. Then for each element r of the range of the mapping there is exactly one element of the domain that maps to r. We also say that the mapping is *bijective* or *a one-to-one correspondence.*

Properties of mappings

Let $\mathbf{B} = \{a, b, c, d\}$, $\mathbf{A} = \{1, 2, 3\}$, $\mathbf{A}_1 = \{1, 2, 3, 4, 5\}$ and $\mathbf{A}_2 = \{1, 2, 3, 4\}$, and let
$m_1 = [1 \mapsto a, 2 \mapsto b, 3 \mapsto c]$ and $m_2 = [1 \mapsto a, 2 \mapsto b, 3 \mapsto a]$
be mappings from $\mathbf{A}$ to $\mathbf{B}$,
$m_3 = [1 \mapsto a, 2 \mapsto a, 3 \mapsto b, 4 \mapsto c, 5 \mapsto d]$ and
$m_4 = [1 \mapsto d, 2 \mapsto c, 3 \mapsto b, 4 \mapsto a]$ be mappings from $\mathbf{A}_1$ to $\mathbf{B}$ and $\mathbf{A}_2$ to $\mathbf{B}$ respectively.

Then the mapping m_1 is injective but not surjective and hence not bijective; the mapping m_2 is neither injective nor surjective, so also not bijective; the mapping m_3 is surjective but not injective, so not bijective; and the mapping m_4 is both injective and surjective and hence also bijective.

Permutations

- *def:* A *permutation* of the positive integers $1, 2, \ldots, n$ is a bijection from $\mathbf{N}_n$ to itself where $\mathbf{N}_n = \{i \in N \mid 1 \leq i \leq n\}$.

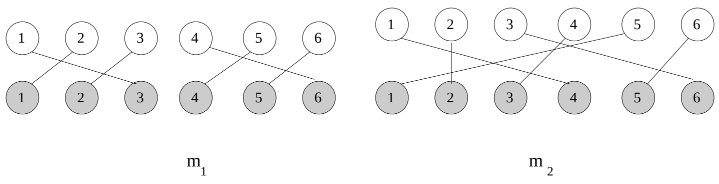

Figure 1: Permutations

- One way to represent a permutation is using a *product of cycles* in which we write all cycles of the permutation without any delimiters. For example, the permutations represented above can be written as $m_1 = (1, 3, 2)(4, 6, 5)$ and $m_2 = (1, 4, 3, 6, 5)(2)$.

Product of mappings (1)

- *def:* Let m_1 and m_2 be two mappings such that $rng(m_1) \subseteq dom(m_2)$. The *product* (or *composition*) of the two mappings is the mapping which maps each element s of the domain $dom(m_1)$ to the value which is the image of $m_1(s)$ under the mapping m_2, i.e. to $m_2(m_1(s))$. It is denoted by $m_1 * m_2$.
  Example: The product of the two mappings $m_1 = (1,3,2)(4,6,5)$ and $m_2 = (1,4,3,6,5)(2)$ is $m_1 * m_2 = (1,6)(2,4,5,3)$.

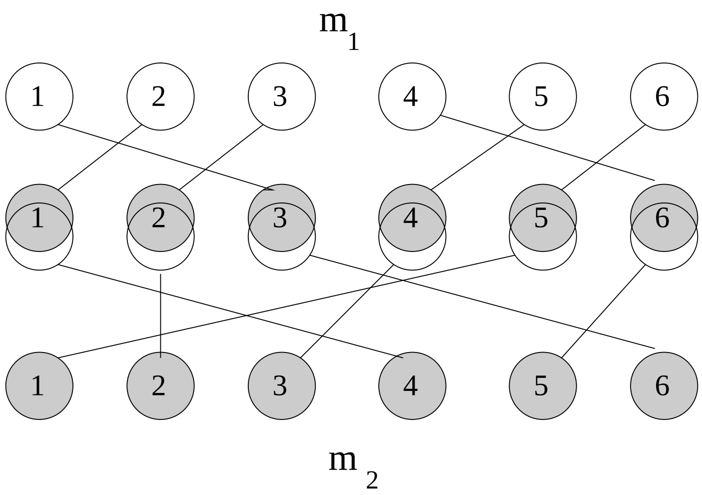

Figure 2: The product of two mappings

n-fold composition of a mapping

Let m be a mapping, the range of which is a subset of its domain, and let $n \in \{i \in \mathbf{N} \mid i \geq 1\}$.

- *def:* $m\hat{\ }n$ is the n-fold composition of the mapping m.
  Example: If $m_1 = (1, 3, 2)(4, 6, 5)$, then $m_1\hat{\ }3 = (1)(2)(3)(4)(5)(6)$.

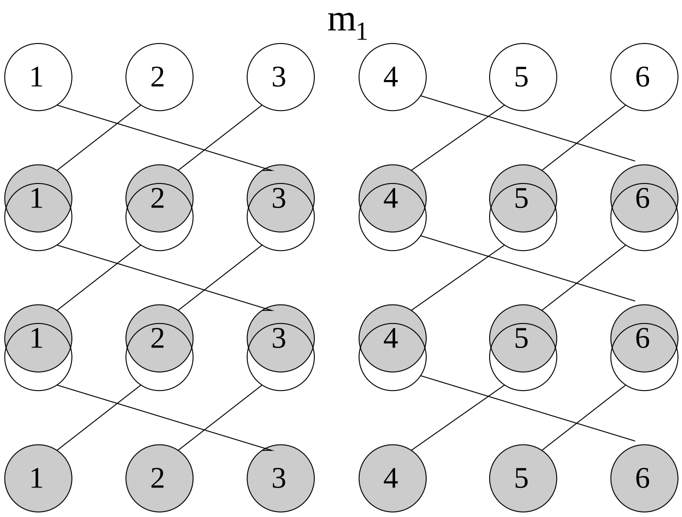

Figure 3: 3-fold composition of a mapping

Override operator

- *def:* The *override* operator overrides one mapping with another where priority is given to the associations in the second operand when the domain values match. It is written $m_1 \dagger m_2$
  where m_1 and m_2 are mappings.

  Example: Let
  $m_1 = [1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 5]$. Then $m_1 \dagger [2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 7] = [1 \mapsto 3, 2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 7, 5 \mapsto 4, 6 \mapsto 5]$.

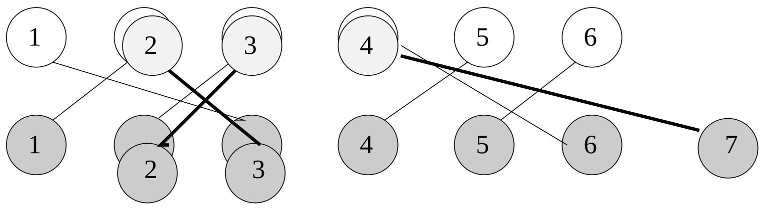

Figure 4: Overriding of mappings

Operations on mappings (4)

- *def:* The *union* operator combines two mappings whose domains are disjoint by constructing the set union of the associations in the two mappings. It is written $m_1 \cup m_2$ where m_1 and m_2 are mappings.

  Example: $[1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2] \cup [4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 5] = [1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 5]$

- *def:* The *domain subtraction* operator removes all the associations from a mapping whose domain values are in a given set. It is written $m \setminus s$ where m is a mapping and s is a set.

  Let $m = [1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 5]$.
  Then $m \setminus \{2, 3, 4, 7\} = [1 \mapsto 3, 5 \mapsto 4, 6 \mapsto 5]$.

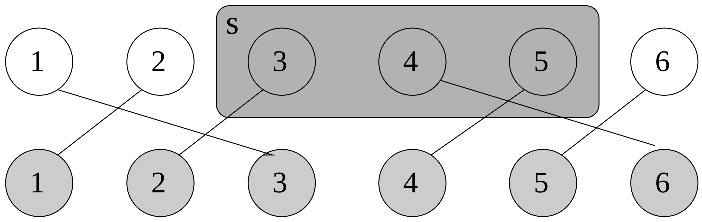

Figure 5: Restricting a mapping by a given set

Domain restriction

- *def:* The *domain restriction* operator removes all the associations from a mapping whose domain values are not in a given set. It is written $m \,/\, s$ where m is a mapping and s is a set.

  Example

  Let $m = [1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 5]$.
  Then $m \,/\{2, 3, 4, 7\} = [2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6]$.

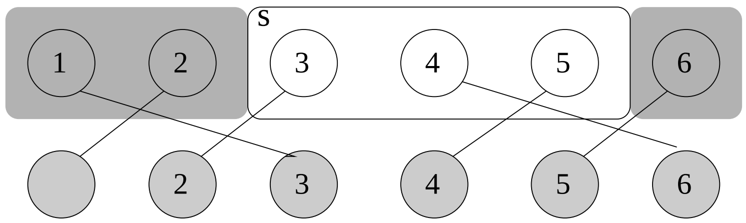

Figure 6: Restricting a mapping to a given set

Operations on mappings (6)

- *def:* The *range subtraction* operator removes all the associations from a mapping whose range values are in a given set. It is written $m \ominus s$ where m is a mapping and s is a set.

  Example: Let $m = [1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 5]$. Then $m \ominus \{2, 3, 4, 7\} = [2 \mapsto 1, 4 \mapsto 6, 6 \mapsto 5]$.

- *def:* The *range restriction* operator removes all the associations from a mapping whose range values are not in a given set. It is written $m \oslash s$ where m is a mapping and s is a set.

  Example: Let $m = [1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 5]$. Then $m \oslash \{2, 3, 4, 7\} = [1 \mapsto 3, 3 \mapsto 2, 5 \mapsto 4]$.