Predicate Logic

- There are some sentences that we cannot tell to be true or false simply by their form. For example:

  There is an even prime number and all prime numbers are not even and
  All men are mortal and Socrates is a man, therefore Socrates is mortal

- The language and methods of propositional logic are not enough to express them fully or to prove their validity.
- The reason is that the language of propositional logic is too primitive to express properties of an object or relationships between objects.
- The language and methods of predicate logic extend propositional logic by enabling us to speak about objects, their properties and the relationships between them and thus provide a way of reasoning about this type of sentence.

Unary predicates

- Let $\mathbf{D}$ be a set and x be a variable which takes some value from the set $\mathbf{D}$.
  def: A unary predicate defined on the set $\mathbf{D}$ is any assertive sentence involving the variable x, and is denoted by $p(x)$, $q(x)$, $r(x)$, etc.
  The set $\mathbf{D}$ is called the domain of the variable x.

Examples:

1. Let $\mathbf{D}$ be the set of all people.
   $p(y)$: 'y is mortal' and $q(y)$: 'y is wise'
   are predicates on $\mathbf{D}$.
2. Let $\mathbf{R}$ be the set of all real numbers.
   $p_1(x)$: '$x \leq 3$' and $q_1(x)$: '$(x \leq -2) \vee (2 \leq x)$'
   are predicates on $\mathbf{R}$.
3. Let $\mathbf{N}$ be the set of all natural numbers.
   $p_2(n)$: 'n is an even number'
   and
   $q_2(n)$: '$2^{2^n} + 1$ is a prime number'
   are predicates on $\mathbf{N}$.

Predicates

- Predicates themselves are not propositions.
- They instead define properties of their subjects, which are represented abstractly by the variables they contain.
- In the predicate $p(y)$: 'y is mortal', the variable y is the subject and '*is mortal*' is its property.
- The value of a predicate depends on the values of its variables.
- When specific values are substituted for the variables the predicate evaluates to either true or false, i.e. it becomes a proposition.

Examples:

- $p_1(\pi)$: '$\pi \leq 3$' = F and
  $q_1(\pi)$: '$(\pi \leq -2) \vee (2 \leq \pi)$' = T
  but $p_1(1)$: '$1 \leq 3$' = T.
- $p_2(12)$: '12 is an even number' = T and
  $q_2(2)$: '$2^{2^2} + 1$ is a prime number' =T
  ($2^{2^2} + 1 = 2^4 + 1 = 16 + 1 = 17$ is prime).

Binary predicate

- Let **D** and **L** be sets and x and y be variables which take their values from the sets **D** and **L** respectively.
  def: A binary predicate defined on the sets **D** and **L** is any assertive sentence involving the variables x and y, and is denoted by $p(x, y)$, $q(x, y)$, $r(x, y)$, etc.
  The sets **D** and **L** are called the domains of the variables x and y respectively.

Examples

1. Let x and y be variables which take values from **R** (the set of all real numbers). $p(x, y)$: '$x \leq y$' and
   $q(x, y)$: '$x^2 + y^2 \geq 4$'
   are binary predicates defined on **R**.
2. Let **D** be the set of all people and x and y be variables taking values from **D**.
   $p_1(x, y)$: 'x and y are roommates'
   and
   $q_1(x, y)$: 'x is taller than y'
   are binary predicates defined on **D**.

Truth sets

def: A predicate is an assertive sentence containing a specific number of variables. It becomes a proposition when specific values taken from the domain of the variables are substituted in place of the variables.

- A predicate containing n ($n \geq 1$) variables is also called an n-ary predicate.

def: The *truth set* of the predicate $p(x)$ is the set of all elements x of its domain which make $p(x)$ true.

We denote the truth set of $p(x)$ by $\mathbf{P}$.

$$\mathbf{P} = \{x \in \mathbf{D} \mid p(x)\}$$

Connections of Predicates (1)

- Now we define operators for predicates. Let $p(x)$ and $q(x)$ be predicates defined on the same domain $\mathbf{D}$.
- **NOT:** The negation of $p(x)$ is the predicate defined on $\mathbf{D}$ which is true only for those elements x of $\mathbf{D}$ for which $p(x)$ is false. It is denoted by $\sim p(x)$ and its truth set is denoted by $\overline{\mathbf{P}}$.

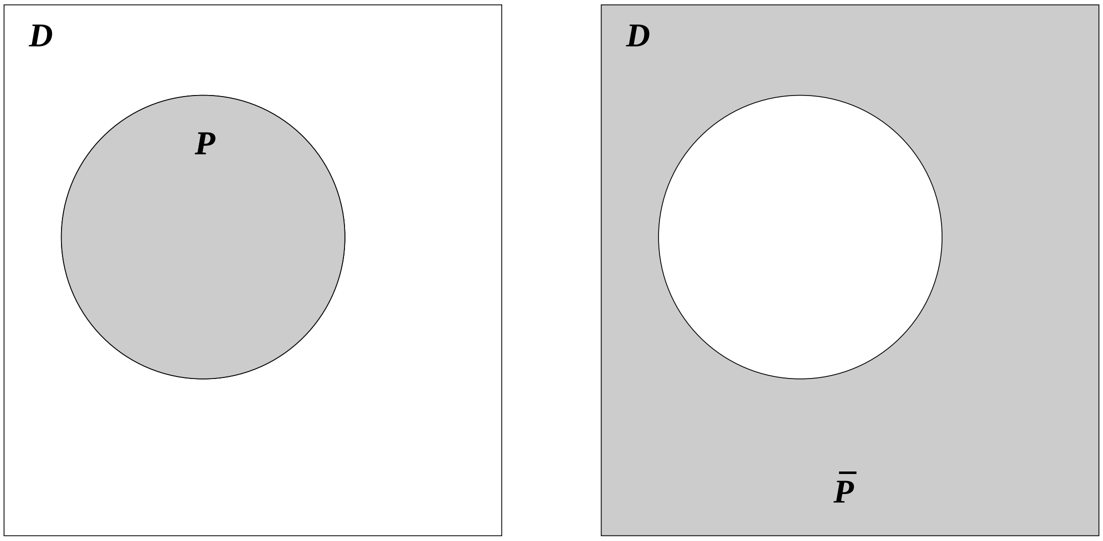

Figure 1: The truth sets of predicates $p(x)$ and $\sim p(x)$

Connections of Predicates (2)

- **AND:** The conjunction $p(x) \wedge q(x)$ is the predicate defined on $\mathbf{D}$ which is true only for those elements x of $\mathbf{D}$ for which both $p(x)$ and $q(x)$ are true.
- **OR:** The disjunction $p(x) \vee q(x)$ is the predicate defined on $\mathbf{D}$ which is true for those elements x of $\mathbf{D}$ for which either $p(x)$ or $q(x)$ or both are true.

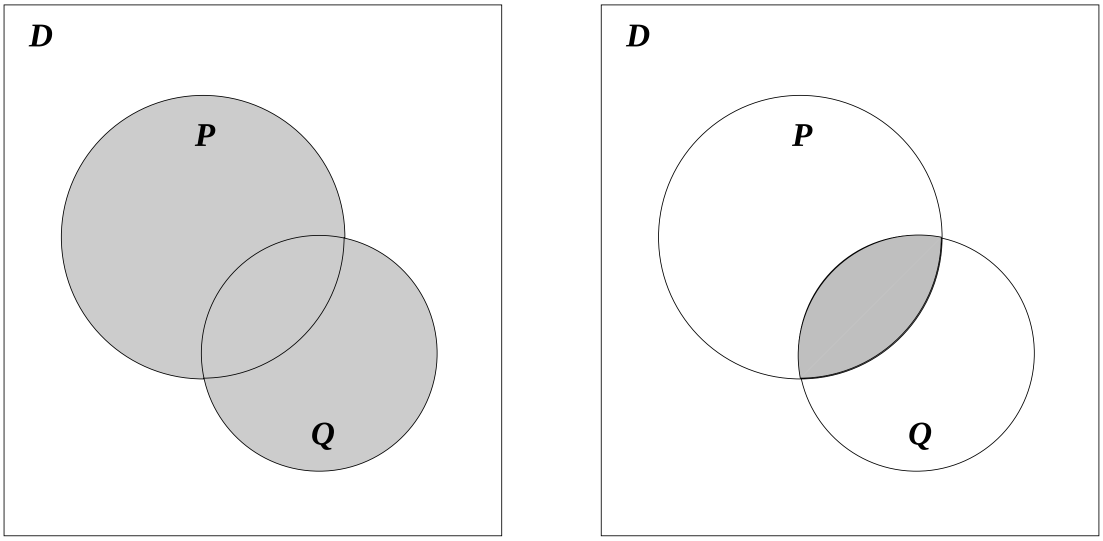

Figure 2: The truth sets of $p(x) \vee q(x)$ and $p(x) \wedge q(x)$

Connections of Predicates (3)

- **CONDITIONAL:** The conditional $p(x) \rightarrow q(x)$ is the predicate defined on $\mathbf{D}$ which is false only for those elements x of $\mathbf{D}$ for which $p(x)$ is true and $q(x)$ is false. It is equivalent to the predicate $\sim p(x) \vee q(x)$.
- **EQUIVALENCE:** The equivalence $p(x) \equiv q(x)$ is the predicate defined on $\mathbf{D}$ which is true for all those elements x of $\mathbf{D}$ for which $p(x)$ and $q(x)$ have the same truth value.

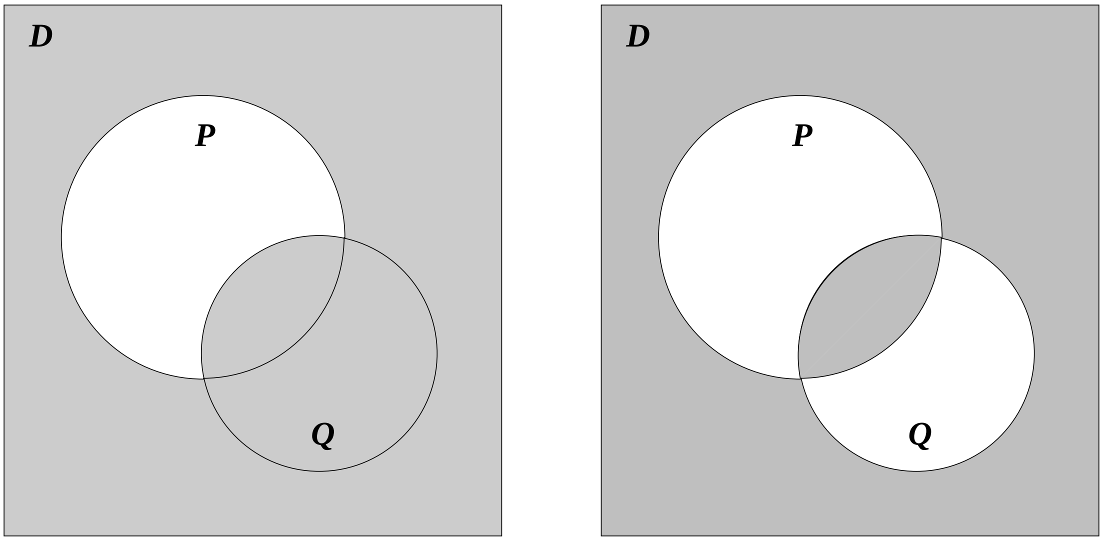

Figure 3: The truth sets of $p(x) \rightarrow q(x)$ and $p(x) \equiv q(x)$

Quantifiers

- We are often interested in statements which indicate the number of elements for which a particular predicate is true:
  1. $p(x)$ is true for all values of x,
  2. $p(x)$ is true for at least one value of x,
  3. $p(x)$ is true for exactly one value of x.
- We write these statements as $\forall x p(x)$, $\exists x p(x)$ and $\exists! x p(x)$ respectively.
- The string $\forall x$ is called *universal quantification*, and $\exists x$ and $\exists! x$ are called *existential quantification.*
- The predicate $p(x)$ appearing in the statements $\forall x p(x)$, $\exists x p(x)$ and $\exists! x p(x)$ is said to be the *scope* of the corresponding quantifier.
- Predicates are not propositions. However the quantifiers change predicates into propositions. This process is called quantification.

Universal and existential statements

- *def:* A universal statement of the form $\forall x p(x)$ containing a predicate is a proposition that is true if, and only if, the predicate is true for every value of its variable within the given domain.
- *def:* An existential statement of the form $\exists x p(x)$ containing a predicate is a proposition that is true if there is at least one value within the variable's domain for which the predicate is true.
- *def:* An existential statement of the form $\exists ! x p(x)$ containing a predicate is a proposition that is true if there is exactly one value within the variable's domain for which the predicate is true.

Examples of quantified statements

- Let $\mathbf{B}$ be the set of all species of non-extinct birds, and x be a variable such that $x \in \mathbf{B}$. Let $q(x)$ be the predicate 'x can fly'.
  The universal statement $\forall x\, q(x)$ corresponds to the statement that all species of non-extinct birds can fly.
  It is obvious that $\forall x\, q(x)$ is false and $\exists x \sim q(x)$ is true.
- Let $n \in \mathbf{N}$ and $p(n)$: '$2^{2^n} + 1$ is a prime number'. Is the statement $\forall n\, p(n)$ true?
  $p(1)$, $p(2)$ and $p(3)$ are true (the numbers 5, 17 and 257 are prime). But checking all numbers in this way is impossible.
  Euler found that $p(5)$ is false (4294967297 is not prime).
  Thus, $\forall n\, p(n)$ is false and $\exists n p(n)$ is true.
- Let $x \in \mathbf{Z}$ (the set of all integers) and let $p(x)$: '$2x^2 - 8x + 7 \leq 0$'. Is the statement $\exists x p(x)$ true?
  The predicate is equivalent to: '$2(x-2)^2 - 1 \leq 0$'.

  Hence, $p(2)$ is true. Therefore, $\exists x p(x)$ is true.

Negation of quantified statements

Let $p(x)$ be a predicate defined on $\mathbf{D}$.

- **The negation of the universal statement** $\forall x p(x)$ can be formed in two ways:

  1. $\sim (\forall x p(x))$, which means '$p(x)$ is not true for all x'.
  2. The last statement is logically equivalent to the statement 'for at least one x, $p(x)$ is not true'. This is equivalent to $\exists x(\sim p(x))$.

  So we obtain the formula: $\sim (\forall x p(x)) = \exists x(\sim p(x))$.

- **The negation of the existential statement** $\exists x p(x)$ can be formed in two ways:

  1. $\sim (\exists x p(x))$, which means 'there does not exist x for which $p(x)$ is true'.
  2. The last is logically equivalent to the statement 'for all x, $p(x)$ is not true'. This is equivalent to $\forall x(\sim p(x))$.

  So we obtain the formula: $\sim (\exists x p(x)) = \forall x(\sim p(x))$.

Multiple quantifiers

- *def:* If a variable v is in the scope of a quantifier then it is said to be a *bound variable* or *bound by the quantifier.* On the other hand, a variable which is not in the scope of any quantifier is said to be *free.*

  Example In the statement $\forall x(p(x, y) \wedge q(x))$ the variable x is bound and y is a free variable.

- When one of the variables in an n-ary $(n \geq 2)$ predicate is quantified, we obtain a new $n - 1$ -ary predicate. Therefore, we can quantify any of the variables in this $n - 1$ -ary predicate, i.e. we can have more than one quantifier in a statement.

Example of multiple quantifiers

- Let $p(x, y)$ be a predicate, where $x \in D_1$ and $y \in D_2$. Then we can construct unary predicates:

Predicates	Domain
$q(x) : \forall y\, p(x, y)$	D_1
$r(x) : \exists y\, p(x, y)$	D_1
$s(y) : \forall x\, p(x, y)$	D_2
$t(y) : \exists x\, p(x, y)$	D_2

Therefore, we can write the following statements:

$$\forall x q(x) = \forall x \forall y\, p(x, y) \qquad \exists x q(x) = \exists x \forall y\, p(x, y)$$
$$\forall x r(x) = \forall x \exists y\, p(x, y) \qquad \exists x r(x) = \exists x \exists y\, p(x, y)$$
$$\forall y s(y) = \forall y \forall x\, p(x, y) \qquad \exists y s(y) = \exists \forall x\, p(x, y)$$
$$\forall y t(y) = \forall y \exists x\, p(x, y) \qquad \exists y t(y) = \exists y \exists x\, p(x, y)$$

Properties of multiple quantifiers

- $\forall x \forall y\, p(x,y) \;=\; \forall y \forall x\, p(x,y)$
  $\exists x \exists y\, p(x,y) \;=\; \exists y \exists x\, p(x,y)$
- $\exists x \forall y\, p(x,y) \;\neq\; \forall y \exists x\, p(x,y)$
- $\sim \forall x \exists y\, p(x,y) \;=\; \exists x \forall y \sim\; p(x,y)$
  Proof:
  $$\begin{aligned} \sim \forall x \exists y\, p(x,y) \quad &= \quad \sim \forall x\, (\exists y\, p(x,y)) \\ &= \quad \exists x \sim (\exists y\, p(x,y)) \\ &= \quad \exists x (\forall y \sim\; p(x,y)) \end{aligned}$$
- $\sim \exists x \forall y\, p(x,y) \;=\; \forall x \exists y \;\sim p(x,y).$

Propositions of predicate logic

def: The propositions of predicate logic are intended to represent relationships between objects.

1. The truth constants
   true and *false* are propositions.

2. The propositional letters
   $P, Q, R, S, P_1, Q_1, R_1, S_1, P_2, ...$ are propositions.

3. If $c_1, c_2, \ldots, c_n$ are constants, where $n \geq 1$, and p is an n -ary predicate, then

   $$p(c_1, c_2, \ldots, c_n)$$

   is a proposition.

Expressions of predicate logic

def: The *expressions* of predicate logic are built from its propositions according to the following rules:

1. Every proposition is an expression.
2. If $v_1, v_2, \ldots, v_n$ are variables or constants, and p is an n -ary predicate, then

$$p(v_1, v_2, \ldots, v_n)$$

   is an expression.
3. If $\mathcal{F}$ is an expression, then so is its negation $\sim \mathcal{F}$
4. If $\mathcal{F}$ and $\mathcal{G}$ are expressions, then so are the following connections of them:

$$\mathcal{F} \wedge \mathcal{G} \qquad \mathcal{F} \vee \mathcal{G} \qquad \mathcal{F} \rightarrow \mathcal{G}$$
$$\mathcal{F} \equiv \mathcal{G}$$

5. If v is any variable and $\mathcal{F}$ is an expression, then $\forall v \mathcal{F}$, $\exists v \mathcal{F}$ and $\exists ! v \mathcal{F}$ are expressions.

Validity of expressions

- *def:* An expression is *closed* if it has no free occurrence of any variable.

Example:
The expression

$$\forall x(p(x,y) \wedge \exists y q(y,z))$$

is not closed. On the other hand, the following expression is closed:

$$\forall x \exists y p(x,y)$$

- *def:* A closed expression $\mathcal{F}$ is valid if it is true under every value of its components.

Establishing validity

- We can use informal methods and common sense to convince ourselves that expressions are valid.

Example: Suppose we want to show the validity of the expression

$$\sim (\forall x p(x)) \equiv \exists x (\sim p(x)).$$

1. $\quad \sim (\forall x p(x)) = \mathrm{T}$

   iff $\forall x p(x) = \mathrm{F}$ *by the definition of* $\sim$

   iff $\exists x (\sim p(x)) = \mathrm{T}$ *by the definition of* $\forall$

2. $\quad \sim (\forall x p(x)) = \mathrm{F}$

   iff $\forall x p(x) = \mathrm{T}$ *by the definition of* $\sim$

   iff $\exists x (\sim p(x)) = \mathrm{F}$ *by the definition of* $\forall$

The two expressions have the same truth value, thus they are equivalent.