Classical Logic

- Logic is the science of order and form. The truth of these sentences can be determined from their structure.
  - There is a zoo in Macau *or* There is not a zoo in Macau
  - $\pi > 1.0$ *or* $\pi \leq 1.0$
  - P *or* (not P)
- Logic helps us to analyse an argument's form to determine if the truth of the conclusion follows from the truth of the preceding statements.
  - If x is a real number such that $x < -2$ or $2 < x$, then $x^2 > 4$.
    Therefore, if $x^2 < 4$, then $-2 < x$ and $x < 2$.
- Logic provides a notation for specifications that is precise, comprehensive, economical and easy to manipulate.

Propositions

def: A *proposition* (or a *statement*) is a declarative sentence which can be classified as true or false, but not both.

- '*I am in Macau*' is a proposition
- '*I am lying*' is not a proposition because it is not possible to say whether it is true or false.
- The sentences of propositional logic are made up of the following symbols, called propositions:
  - The truth constants

    true (T) and *false* (F)

  - The propositional letters
    $P, Q, R, S, P_1, Q_1, R_1, S_1, P_2, ...$

Logical Connectives (1)

Logical connectives (or *logical operators*) are used to build compound sentences from propositions.

Let P and Q be statements.

- **AND**
  - The proposition 'P and Q', denoted by $P \wedge Q$, is true if both P and Q are true and false otherwise.
  - It is called the conjunction of P and Q.
    1. '2+2=4' $\wedge$ '3−2=1' is true.
    2. 'It is raining' $\wedge$ 'It is not raining' is false.
- **OR**
  - The proposition 'P or Q', denoted by $P \vee Q$, is false only if both P and Q are false; otherwise it is true.
  - It is called the disjunction of P and Q.
    1. '2+2=4' $\vee$ 'I am sitting' is true.
    2. 'It is raining' $\vee$ 'It is not raining' is true.
    3. '$9 - 1 < 7$' $\vee$ '$10 - 1 < 8$' is false.

Logical Connectives (2)

- **NOT**
  - The proposition 'not P', denoted by $\sim P$, is true if P is false and false if P is true.
  - $\sim P$ is called the negation of P.
  - P is true if $\sim P$ is false and conversely.
  - Examples
    1. not '2+2=4' (or $\sim$ '2+2=4') is false.
    2. $\sim (9-1<7 \ \vee \ 10-1<8)$ is true.
- **CONDITIONAL**
  - A conditional statement is a statement of the form 'if P, then Q'. It is written as $P \rightarrow Q$.
  - $P \rightarrow Q$ is false only if P is true and Q is false, otherwise it is true.
    1. The sentence 'If today is Friday, then 3+2=6' would be written in the form $P \rightarrow Q$.
    2. 'If it is sunny today, then we will go to the beach' is written in the form $P \rightarrow Q$ also.

Logical Connectives (3)

- **EQUIVALENCE** or **BICONDITIONAL**
  - A statement of the form: 'P if and only if Q' is called the equivalence or the biconditional of P and Q. It is written $P \equiv Q$ or $P \leftrightarrow Q$ or P iff Q.
  - $P \equiv Q$ is true only if P and Q have the same truth value.
    1. '1+1=2' $\equiv$ '5−3=2' is true.
    2. 'The earth is flat' $\equiv$ '3 $<$ 5' is false.

Logical Connectives (4)

- The truth values of the connectives are summarised in the following *truth tables*:

P	$\sim P$
T	F
F	T

P	Q	$P \wedge Q$	$P \vee Q$	$P \to Q$	$P \equiv Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Propositional Expressions

- Expressions of propositional logic are built up from propositions by application of the propositional connectives.
- Expressions are formed according to the following rules:
  - Every proposition, i.e. a truth constant or a propositional letter, is an expression.
  - If $\mathcal{F}$ and $\mathcal{G}$ are expressions, then so are the following:

$$(\sim \mathcal{F})$$
$$(\sim \mathcal{G})$$
$$(\mathcal{F} \wedge \mathcal{G})$$
$$(\mathcal{F} \vee \mathcal{G})$$
$$(\mathcal{F} \rightarrow \mathcal{G})$$
$$(\mathcal{F} \equiv \mathcal{G})$$

Precedence rules

1. The order of evaluation is $\sim \wedge \vee \rightarrow \equiv$.
2. We may always omit the outermost pair of parentheses.
3. The connective which has higher priority than all others in an expression evaluates first, and occurrences of the same connective evaluate from left to right.
4. If there are more than one successive occurrence of the same connective, then the parentheses of the subexpression on the left side of each connective may be omitted.
5. We may omit all pairs of parentheses which can unambiguously be restored using the above four rules.

Examples of Expressions

Let P and Q be propositions. Then the following are expressions:

- $(P \vee Q)$, $(\sim P)$, and $(\sim Q)$
- $(\sim (P \vee Q))$ and $((\sim P) \wedge (\sim Q))$
- $((\sim (P \vee Q)) \equiv ((\sim P) \wedge (\sim Q)))$
- These expressions can be written as:

$$P \vee Q, \qquad \sim P, \qquad \text{and} \qquad \sim Q$$

$$\sim (P \vee Q) \qquad \text{and} \qquad \sim P \wedge \sim Q$$

$$\sim (P \vee Q) \equiv \sim P \wedge \sim Q$$

Additional Connectives (1)

- In computer science, the three following connectives are often used.

1. XOR

- The $\vee$ connective is "inclusive" in the sense that it is true in the case when both P and Q are true.
- But, in the statement '*Today is 5 or 6 of June*' the *or* connective should be "exclusive".
- The Exclusive OR of P and Q, denoted by $PxorQ$, is the proposition that is true when exactly one of P and Q is true and false otherwise.
- $PxorQ = (P \vee Q) \wedge \sim (P \wedge Q)$ holds for xor.

Additional Connectives (2)

2. NOR

- P *nor* Q is a proposition which is defined by the expression $\sim (P \vee Q)$, i.e.

$$P \text{ } nor \text{ } Q = \sim (P \vee Q).$$

- Equivalently, P *nor* $Q = \sim P \wedge \sim Q$.

3. NAND

- P *nand* Q is a proposition which is defined by the expression $\sim (P \wedge Q)$, i.e.

$$P \text{ } nand \text{ } Q = \sim (P \wedge Q)$$

- Equivalently, P *nand* $Q = \sim P \vee \sim Q$.

Additional Connectives (3)

The truth values of these three connectives are:

P	Q	P *xor* Q	P *nor* Q	P *nand* Q
T	T	F	F	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	T

Properties of expressions

1. *def:* An expression $\mathcal{E}$ is called a *tautology* if it is always true regardless of the truth values of its propositional letters.

2. *def:* A *contradiction* is an expression which is false regardless of the truth values of its propositional letters.

   - If $\mathcal{E}$ is a tautology, then $\sim \mathcal{E}$ is a contradiction and vice versa.

3. *def:* An expression is *valid* if it is a tautology.

4. *def:* An expression $\mathcal{E}$ *implies* an expression $\mathcal{F}$ if $(\mathcal{E} \to \mathcal{F})$ is a tautology. It is written $\mathcal{E} \Rightarrow \mathcal{F}$.

5. *def:* Expressions $\mathcal{E}$ and $\mathcal{F}$ are called *logically equivalent* if $(\mathcal{E} \equiv \mathcal{F})$ is a tautology. It is written $\mathcal{E} = \mathcal{F}$.

Tautology and Contradiction

- The most straightforward way to determine whether an expression is a tautology or a contradiction is by a complete analysis of all possible values of the expression using its truth table.

Examples:

$$\mathcal{E}\colon P \wedge Q \to P$$
$$\mathcal{F}\colon P \wedge \sim P \wedge Q$$

The corresponding truth tables are

		1	$\mathcal{E}$	*2*	*1*	$\mathcal{F}$
P	Q	$P \wedge Q$	$\to P$	$P \wedge$	$\sim P$	$\wedge Q$
T	T	T	T	F	F	F
T	F	F	T	F	F	F
F	T	F	T	F	T	F
F	F	F	T	F	T	F

Logical Equivalences (1)

- Tautologies and logically equivalent expressions are very useful for making the analytic transformations necessary to simplify complex expressions and to prove other tautologies or establish the equivalence of other expressions.
- Basic Tautologies
  $\mathcal{F} \vee (\sim \mathcal{F}) = \mathrm{T}$
  $\mathcal{F} \rightarrow \mathcal{F} = \mathrm{T}$
  $\mathcal{F} \equiv \mathcal{F} = \mathrm{T}$
  $(\mathcal{F} \wedge \mathcal{G}) \rightarrow \mathcal{F} = \mathrm{T}$
- Identity Laws
  $\mathcal{F} \wedge \mathrm{T} = \mathcal{F}$
  $\mathcal{F} \vee \mathrm{F} = \mathcal{F}$
- Domination Laws
  $\mathcal{F} \vee \mathrm{T} = \mathrm{T}$
  $\mathcal{F} \wedge \mathrm{F} = \mathrm{F}$
- Double negation Law
  $\sim (\sim \mathcal{F}) \ = \mathcal{F}$

Logical Equivalences (2)

- Commutative Laws
  $\mathcal{F} \vee \mathcal{G} = \mathcal{G} \vee \mathcal{F}$
  $\mathcal{F} \wedge \mathcal{G} = \mathcal{G} \wedge \mathcal{F}$
  $\mathcal{F} \equiv \mathcal{G} = \mathcal{G} \equiv \mathcal{F}$

- Associative Laws
  $(\mathcal{F} \vee \mathcal{G}) \vee \mathcal{H} = \mathcal{F} \vee (\mathcal{G} \vee \mathcal{H})$
  $(\mathcal{F} \wedge \mathcal{G}) \wedge \mathcal{H} = \mathcal{F} \wedge (\mathcal{G} \wedge \mathcal{H})$
  $(\mathcal{F} \equiv \mathcal{G}) \equiv \mathcal{H} \ = \ \mathcal{F} \equiv (\mathcal{G} \equiv \mathcal{H})$

- Distributive Laws
  $\mathcal{F} \vee (\mathcal{G} \wedge \mathcal{H}) = (\mathcal{F} \vee \mathcal{G}) \wedge (\mathcal{F} \vee \mathcal{H})$
  $\mathcal{F} \wedge (\mathcal{G} \vee \mathcal{H}) = (\mathcal{F} \wedge \mathcal{G}) \vee (\mathcal{F} \wedge \mathcal{H})$

- DeMorgan's Laws
  $\sim (\mathcal{F} \wedge \mathcal{G}) =\sim \mathcal{F} \vee \sim \mathcal{G}$
  $\sim (\mathcal{F} \vee \mathcal{G}) =\sim \mathcal{F} \wedge \sim \mathcal{G}$
  $\sim (\mathcal{F} \rightarrow \mathcal{G}) \ = \ \mathcal{F} \wedge \sim \mathcal{G}$
  $\sim (\mathcal{F} \equiv \mathcal{G}) = (\mathcal{F} \equiv\sim \mathcal{G})$

Logical Equivalences (3)

- Transitive Laws
  $(\mathcal{F} \rightarrow \mathcal{G}) \wedge (\mathcal{G} \rightarrow \mathcal{H}) \rightarrow (\mathcal{F} \rightarrow \mathcal{H})$
  $(\mathcal{F} \equiv \mathcal{G}) \wedge (\mathcal{G} \equiv \mathcal{H}) \rightarrow (\mathcal{F} \equiv \mathcal{H})$

- *modus ponens* Law
  $\mathcal{F} \wedge (\mathcal{F} \rightarrow \mathcal{G}) \rightarrow \mathcal{G}$

- *modus tollens* Law
  $(\mathcal{F} \rightarrow \mathcal{G}) \wedge \sim \mathcal{G} \rightarrow \sim \mathcal{F}$

- Tautologies for Eliminating Connectives
  $\mathcal{F} \rightarrow \mathcal{G} = \sim \mathcal{F} \vee \mathcal{G}$
  $\mathcal{F} \rightarrow \mathcal{G} = \sim (\mathcal{F} \wedge \sim \mathcal{G})$
  $\mathcal{F} \vee \mathcal{G} = (\sim \mathcal{F} \rightarrow \mathcal{G})$
  $\mathcal{F} \wedge \mathcal{G} = \sim (\mathcal{F} \rightarrow \sim \mathcal{G})$
  $\mathcal{F} \equiv \mathcal{G} = (\mathcal{F} \rightarrow \mathcal{G}) \wedge (\mathcal{G} \rightarrow \mathcal{F})$
  $\mathcal{F} \equiv \mathcal{G} = (\mathcal{F} \wedge \mathcal{G}) \vee (\sim \mathcal{F} \wedge \sim \mathcal{G})$

Proving Examples (1)

- Prove that $\sim(\sim\mathcal{F}) \to \mathcal{F} \wedge (\mathcal{F} \vee \mathcal{G})$ is a tautology:

$$
\begin{array}{rll}
 & \text{Expressions} & \\
\hline
 & \sim(\sim\mathcal{F}) \to \mathcal{F} \wedge (\mathcal{F} \vee \mathcal{G}) & = \\
= & \mathcal{F} \to \mathcal{F} \wedge (\mathcal{F} \vee \mathcal{G}) & \\
= & \mathcal{F} \to \mathcal{F} & \\
= & \mathrm{T} &
\end{array}
$$

Proving Examples (2)

- Prove that the expressions $(\sim \mathcal{F} \rightarrow \mathcal{G})$ and $(\sim \mathcal{G} \rightarrow \mathcal{F})$ are equivalent:

$$
\begin{array}{cc}
 & \text{Expressions} \\
\hline
 & \sim \mathcal{F} \rightarrow \mathcal{G} \qquad = \\
= & \sim (\sim \mathcal{F}) \vee \mathcal{G} \\
= & \mathcal{F} \vee \mathcal{G} \\
= & \mathcal{G} \vee \mathcal{F} \\
= & \sim (\sim \mathcal{G}) \vee \mathcal{F} \\
= & (\sim \mathcal{G} \rightarrow \mathcal{F})
\end{array}
$$