Relations

- *def: (n-ary relation)* An *n-ary relation* R between sets A_1, A_2, …, A_n is a subset of $A_1 \times A_2 \times \cdots \times A_n$:

$$\mathrm{R} = \{(x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \cdots \times A_n \mid p(x_1, x_2, \ldots, x_n)\}$$

  where $n \in \mathbf{N}$ ($n \geq 2$), and p is a predicate representing the properties of the elements of the relation R.

- n is called the *arity* of the relation R.

- *def: (binary relation)* A relation R of arity 2 is called a *binary relation* from A_1 to A_2. A_1 is called the *domain* and A_2 the *co-domain* of the relation. If (x, y) is in R, then we denote this by $x\mathrm{R}y$.

- *def: (relation on a set)* If $S = A_1 = A_2 = \cdots = A_n$ for a relation R, then R is called a *relation on the set S*.

Examples of relations (1)

- Binary relations can be represented by arrow diagrams.

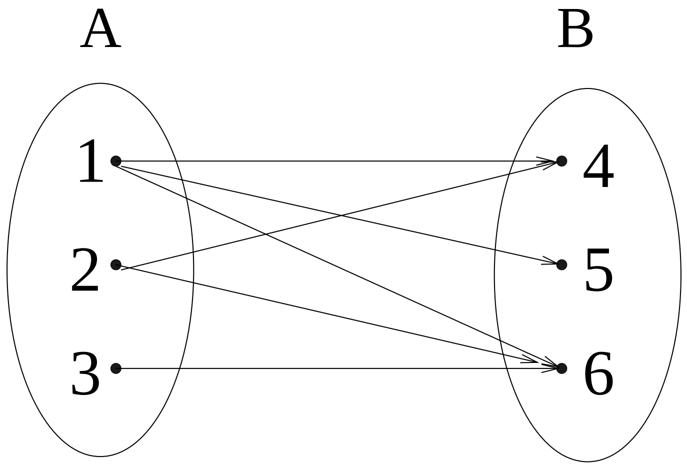


Figure 1: A relation represented by an arrow diagram

- Let $\mathbf{A} = \{1, 2, 3\}$ and $\mathbf{B} = \{4, 5, 6\}$, and let R be a binary relation defined as follows:

$$\{(x, y) \mid y/x \in \mathbf{Z}\}$$

Then $R = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 6), (3, 6)\}$.

Examples of relations (2)

- Let $\mathbf{S} = \{-2, -1, 0, 1, 2\}$, and let R be a relation on $\mathbf{S}$ defined as follows:

$$\{(x, y) \mid x = -y \vee x/y = -2\}$$

Then $R =$
$\{(-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2), (2, -1), (-2, 1)\}$.

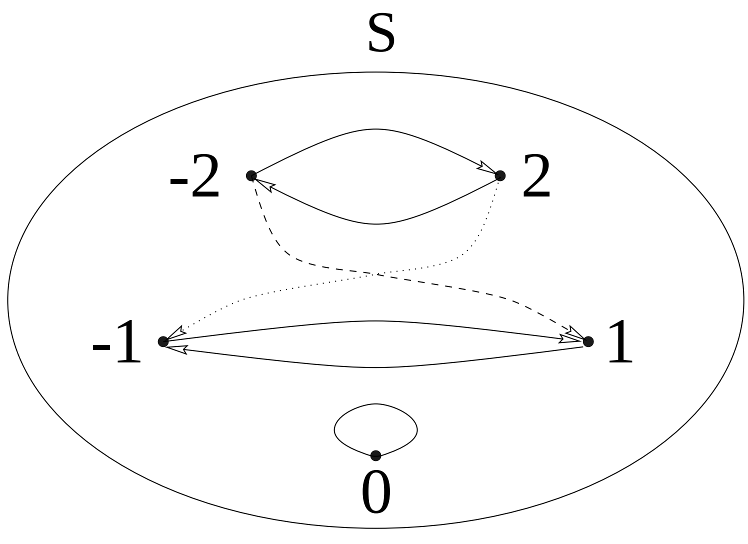


Figure 2: A binary relation on a set

Functions and relations

- *def: (functions in terms of binary relations)* A function f from set $\mathbf{X}$ to set $\mathbf{Y}$ is a relation with two properties:

  1. for every element x in $\mathbf{X}$, there exists an element in $\mathbf{Y}$ such that $(x, y) \in f$;
  2. for all elements x in $\mathbf{X}$ and y and z in $\mathbf{Y}$, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$; i.e. each element x in $\mathbf{X}$ is related to a unique element y in $\mathbf{Y}$.

- Let **Bool** be the set $\{$*true*, *false*$\}$.

  def: (binary relations in terms of functions) A binary relation on a set $\mathbf{S}$ is a function $r : \mathbf{S} \times \mathbf{S} \to \mathbf{Bool}$ such that $r(x, y)$ is *true* for every $(x, y) \in \mathbf{S} \times \mathbf{S}$ for which x and y are related and *false* otherwise.

  Example: Let $\mathbf{S} = \{-2, -1, 0, 1, 2\}$, and let R be a relation on $\mathbf{S}$ defined as:
  $\{(x, y) \mid x = -y \vee x/y = -2\}$

  $$r(x, y) = \text{`}x = -y \vee x/y = -2\text{'}$$

  which is true for each pair in the set
  $\{(-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2), (2, -1), (-2, 1)\}$.

Classification of binary relations (1)

Let r be a binary relation on a set $\mathbf{S}$, i.e.

$$r : \mathbf{S} \times \mathbf{S} \rightarrow \mathbf{Bool}$$

- *def:* r is *reflexive* if and only if $r(x, x)$ is true for every element x in $\mathbf{S}$, i.e. if $\forall x(x \in \mathbf{S} \rightarrow r(x, x))$ then r is reflexive.

- *def:* r is *irreflexive* if and only if $r(x, x)$ is false for every element x in $\mathbf{S}$.

- *def:* r is *symmetric* if and only if $r(x, y) = r(y, x)$ for every x and y in $\mathbf{S}$.

- *def:* r is *antisymmetric* if and only if $r(x, y)$ and $r(y, x)$ are both true only if $x = y$ for every x and y in $\mathbf{S}$.

  For an antisymmetric relation on $\mathbf{S}$, only one of $r(x, y)$ and $r(y, x)$ can be true for all distinct elements x and y in $\mathbf{S}$.

- *def:* r is *transitive* if and only if whenever $r(x, y)$ and $r(y, z)$ are both true, $r(x, z)$ is also true for every x, y and z in $\mathbf{S}$.

Classification of binary relations (2)

Example

Consider the following relations on $\mathbf{Z}$.

$R_1 = \mathbf{Z} \times \mathbf{Z}$

$R_2 = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid x = y\}$

$R_3 = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid x < y\}$

$R_4 = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid x \leq y\}$

$R_5 = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid$ 'x is divisible by y'$\}$

The table below shows the properties of each of these relations:

	R_1	R_2	R_3	R_4	R_5
reflexive	$\surd$	$\surd$		$\surd$	$\surd$
irreflexive			$\surd$		
symmetric	$\surd$	$\surd$			
antisymmetric		$\surd$			
transitive	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

Classification of binary relations (3)

- *def: (equivalence relation)* A binary relation on $\mathbf{S}$ is called an *equivalence relation* on $\mathbf{S}$ if and only if it is reflexive, symmetric and transitive.

  Example: Let n be a positive integer and let R be the relation defined by
  R$= \{(i,j) \in \mathbf{Z} \times \mathbf{Z} \mid \exists k(k \in \mathbf{Z} \wedge k \cdot n = i - j)\}$

  – R is reflexive, since $0 \cdot n = i - i$ for every $i \in \mathbf{Z}$.

  – R is symmetric, since if $k \cdot n = i - j$ then $(-k) \cdot n = j - i$, i.e. if $(i,j) \in$R then $(j,i) \in$R.

  – R is transitive.
  Proof: Suppose that $(i,j) \in$R and $(j,m) \in$R. Then there are integers $k_1$and k_2 such that $k_1 \cdot n = i - j$ and $k_2 \cdot n = j - m$. Thus,

  $$k_1 \cdot n + k_2 \cdot n = (i-j) + (j-m) \text{ or}$$
  $$(k_1 + k_2) \cdot n = (i - m) \text{ where } k_1 + k_2 \text{ is an integer.}$$

  Hence, $(i,m) \in$R.

  Therefore, R is an equivalence relation on $\mathbf{Z}$.

Operations on relations (1)

Let R_1 and R_2 be binary relations from $\mathbf{X}$ to $\mathbf{Y}$.

1. Relations are sets of ordered pairs. Therefore, we can define some operations on relations which are similar to the operations on sets.

   - **Union:** $\mathrm{R}_1 \cup \mathrm{R}_2 = \{(x, y) \in \mathbf{X} \times \mathbf{Y} \mid (x, y) \in \mathrm{R}_1 \vee (x, y) \in \mathrm{R}_2\}$
   - **Intersection:** $\mathrm{R}_1 \cap \mathrm{R}_2 = \{(x, y) \in \mathbf{X} \times \mathbf{Y} \mid (x, y) \in \mathrm{R}_1 \wedge (x, y) \in \mathrm{R}_2\}$
   - **Difference:** $\mathrm{R}_1 - \mathrm{R}_2 = \{(x, y) \in \mathbf{X} \times \mathbf{Y} \mid (x, y) \in \mathrm{R}_1 \wedge (x, y) \notin \mathrm{R}_2\}$
   - **Complement:** $\overline{\mathrm{R}_1} = \mathbf{X} \times \mathbf{Y} - \mathrm{R}_1\}$

2. *def: (inverse)* The *inverse* R^{-1} of a given relation R from $\mathbf{X}$ to $\mathbf{Y}$ is defined by:
   $\mathrm{R}^{-1} = \{(x, y) \in \mathbf{X} \times \mathbf{Y} \mid (y, x) \in \mathrm{R}\}$

Operations on relations (2)

- Let R_1 be a binary relation from $\mathbf{X}$ to $\mathbf{Y}$ and R_2 be a binary relation from $\mathbf{Y}$ to $\mathbf{Z}$.

3. *def:* The *composition* $R_1 \circ R_2$ of the two relations R_1 and R_2 is the relation defined as follows:
$R_1 \circ R_2 = \{(x, z) \in \mathbf{X} \times \mathbf{Z} \mid \exists y((x, y) \in R_1 \wedge (y, z) \in R_2\}$

Example: Let $R_1 = \{(a, a), (a, b), (b, d)\}$ and $R_2 = \{(a, d), (b, c), (b, d), (c, b)\}$ be relations on the set $\{a, b, c, d\}$.

Then $R_1 \circ R_2 = \{(a, d), (a, c)\}$.

Transitive and reflexive closures (1)

- Suppose the set $\mathbf{S} = \{a, b, c, d, e\}$, and a relation R defined on the set $\mathbf{S}$ as follows:

$$\mathrm{R}= \{(a, e), (e, d), (d, c), (e, c), (c, b)\}$$

are given.

- The relation is defined on a single set, an arrow diagram for R becomes a *directed graph.* Instead of creating two regions and mapping from one region to the other we list the elements of $\mathbf{S}$ as points in a single region and draw an arrow between points which are related to each other under R.

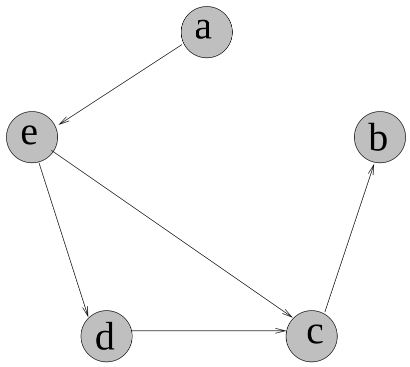
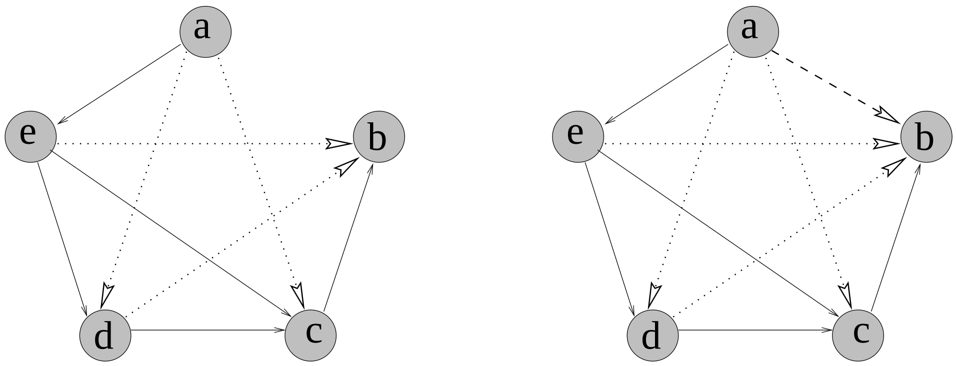


Figure 3: A given intransitive relation

Transitive and reflexive closures (2)

- What associations would have to be added to make this directed graph transitive?
- We would need to add the edges (a, d), (a, c), (e, b) and (d, b) to complete the triangles in this graph. We added associations to the relation R to form a new relation, which we denote by R^+.

$$\mathrm{R}^+ = \{(a,e),(e,d),(a,d),(d,c),\\ (e,c),(a,c),(c,b),(e,b),(d,b)\}$$



(a) R^+ (b) R^t

Figure 4: The transitive closure of the relation

Transitive and reflexive closures (3)

- R^+ is still not transitive. Therefore, more edges must be added to create a transitive graph. The new relation which has been created, which is denoted by R^t, is transitive.

- The relation created in this way is called the *transitive closure* of the relation R.

  The transitive closure of a relation R satisfies the following properties:

  1. R^t is transitive.
  2. R is a subset of R^t, i.e. $R \subseteq R^t$.
  3. If S is any other transitive relation such that $R \subseteq S$, then $R^t \subseteq S$, i.e. the transitive closure R^t is the minimal transitive relation containing the relation R.

Reflexive and transitive closure

- Let R be a relation defined on a set **S**. Then, the *reflexive and transitive closure* of the relation R is the relation R^* defined on **S**, as follows:

$$\mathrm{R}^* = \bigcup_{n \geq 0} \mathrm{R}^n$$

where R^n are binary relations on **S** such that

$$s\mathrm{R}^0 t \text{ iff } s = t$$
$$s\mathrm{R}^n t \text{ iff } \exists u (s\mathrm{R}^{n-1} u \wedge u\mathrm{R} t) \text{ if } n > 0$$

The following properties hold for these relations:

1. $\mathrm{R}^1 = \mathrm{R}$
2. $s\mathrm{R}^* t$ iff $\exists n \geq 0, \exists s_0, \ldots, \exists s_n \in \mathbf{S}$ with $s_0 = s, \ \ s_n = t$, and $s_i \mathrm{R} s_{i+1}$ for all $i < n$.
3. $s\mathrm{R}^* t$ is reflexive and transitive.
4. If T is any reflexive and transitive relation on **S** such that $\mathrm{R} \subseteq T$, then $\mathrm{R}^* \subseteq T$. That is R^* is the smallest reflexive and transitive relation on **S** that contains R.