Sets and elements

- *def:* A *set* is a collection of objects which can be distinguished from one another. The objects are called the *elements* or *members* of the set.
- The following symbols are used:
  - The elements: a, b, c, a_1, b_1, c_1, a_2, b_2, c_2, $\ldots$
  - The sets: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$, $\mathbf{A}_2$, $\ldots$
- *def:* The elements of a set belong to the set or are members of the set. All other objects are not elements of the set and do not belong to the set.
- We write the statements
  - 'the element a belongs to the set $\mathbf{S}$',
  - 'the element b does not belong to the set $\mathbf{S}$'

  symbolically as

$$a \in \mathbf{S} \qquad \text{and} \qquad b \notin \mathbf{S}$$

Defining sets (1)

1. **Enumeration:** In this form, we simply write the elements of the set between braces { }.

   Example: **A**={2, 4, 6, 8} is the set containing the numbers 2, 4, 6 and 8.

   - The order in which its elements appear, or the number of times that a particular element appears has no impact on the set.

     $\mathbf{A} = \{2, 4, 6, 8\} = \{8, 4, 6, 2\} = \{2, 4, 6, 8, 2\}$.
   - There is a difference between a set and the element in the set.

     For example, $\{moon\}$ denotes the set containing only the element *moon*. Consequently, $\{moon\} \neq moon$.

Defining sets (2)

2. Set comprehension: Let $\mathbf{S}$ be a set and $p(x)$ be a predicate defined on the set $\mathbf{S}$. Then the expression

$$\{v \in \mathbf{S} \mid p(v)\}$$

denotes the set of all elements v of $\mathbf{S}$ for which $p(v)$ is true.

Examples:

$\{n \in \mathbf{Z} \mid -2 \leq n < 3\}$	is the set $\{-2, -1, 0, 1, 2\}$
$\{n \in \mathbf{N} \mid 10 \leq n \leq 200\}$	is the set of natural numbers from 10 to 200 inclusive.
$\{x \in \mathbf{R} \mid -10 < x < 10\}$	is the set of real numbers between -10 and 10.
$\{x \in \mathbf{Z} \mid x^2 = 4\}$	is the set $\{-2, 2\}$

The universal and the empty sets

- *def:* The *universal set* is the set of all the elements which make up different sets being considered. Typically, the universal set is represented by **U**.

  Example: If the universe being discussed includes sets made up of rational numbers, integers and natural numbers, then the universal set might be defined to be the set of all real numbers.

- *def:* The *empty set* is the set which contains no elements and is generally written as $\{\}$ or $\emptyset$.
  Example: The set $\{x \in \mathbf{R} \mid x^2 = -1\}$ is empty, i.e.

$$\{x \in \mathbf{R} \mid x^2 = -1\} = \{\}$$

Relationships between sets (1)

Let **A** and **B** be arbitrary sets.

- *def:* **A** is a *subset* of **B**, written $\mathbf{A} \subseteq \mathbf{B}$, if and only if every element of **A** is also an element of **B**.

  We can write this as $\mathbf{A} \subseteq \mathbf{B} \; = \; \forall a(a \in \mathbf{A} \rightarrow a \in \mathbf{B}))$.

- *def:* Two sets **A** and **B** are *equal*, denoted $\mathbf{A}{=}\mathbf{B}$, if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$

- *def:* **A** is a *proper subset* of **B**, written $\mathbf{A} \subset \mathbf{B}$, if and only if **A** is a subset of **B** but **A** is not equal to **B**.

- *def:* If **A** is a subset of **B**, then **B** is a *superset* of **A**, written $\mathbf{B} \supseteq \mathbf{A}$. Likewise, if **A** is a proper subset of **B**, then **B** is a *proper superset* of **A**, written $\mathbf{B} \supset \mathbf{A}$.

Relationships between sets (2)

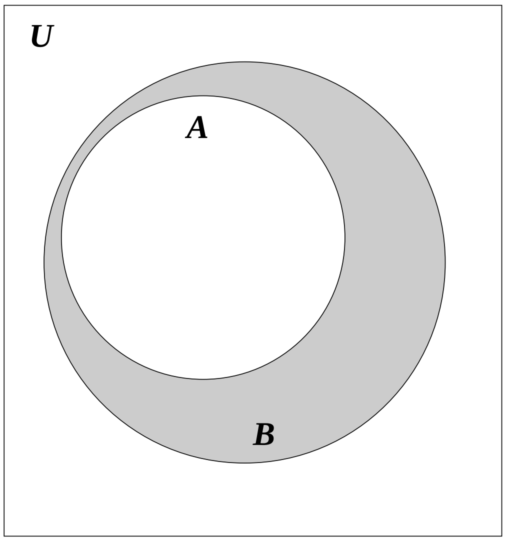

Figure 1: $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{B} \supset \mathbf{A}$

Example Let $\mathbf{A} = \{n \in \mathbf{N} \mid n \leq 6\}$, $\mathbf{B} = \{1, 3, 7\}$, $\mathbf{C} = \{2, 4, 6\}$ and $\mathbf{D} = \{5, 2, 1, 3, 4, 6\}$.

- **B** is not a subset of **A**, i.e. $\mathbf{B} \not\subseteq \mathbf{A}$
- **C** is a subset of **A**, i.e. $\mathbf{C} \subseteq \mathbf{A}$
- **C** is a proper subset of **A**, i.e. $\mathbf{C} \subset \mathbf{A}$
- **A** and **D** are equal, i.e. $\mathbf{A}{=}\mathbf{D}$
- **A** is a proper superset of **C**, i.e. $\mathbf{A}{\supset}\mathbf{C}$

Cardinality and power set

- *def:* The *cardinality of a set* is simply the number of elements in the set. The cardinality of a set $\mathbf{S}$ is denoted by $|\mathbf{S}|$ or *card* $\mathbf{S}$.

  For example: $\mathbf{A} \subset \mathbf{B} \Rightarrow |\mathbf{A}| \leq |\mathbf{B}|$

- *def:* The *power set* of a set $\mathbf{A}$ is the set of all subsets of $\mathbf{A}$. It is written as $P(\mathbf{A})$ or 2^A.

  - For any set $\mathbf{A}$, both $\{\}$ and $\mathbf{A}$ itself are elements of $P(\mathbf{A})$.
  - For any set $\mathbf{A}$ the following equality holds:

    $$|\,P(\mathbf{A})\,| = 2^{|\mathbf{A}|}$$

    Example

    If $\mathbf{A} = \{1, 2, 3\}$ then $P(\mathbf{A}) =$
    $\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Operations on sets (1)

- *def:* The *union* of two sets $\mathbf{A}$ and $\mathbf{B}$ is the set of all elements which are in either $\mathbf{A}$ or $\mathbf{B}$ or both. It is written as $\mathbf{A} \cup \mathbf{B}$.

  $\mathbf{A} \cup \mathbf{B} = \{x \in \mathbf{U} \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$

  - For any set $\mathbf{A}$ the following hold:

    $$\mathbf{A} \cup \{\} = \mathbf{A} \text{ and } \mathbf{A} \cup \mathbf{U} = \mathbf{U}.$$

- *def:* The *intersection* of two sets $\mathbf{A}$ and $\mathbf{B}$ is the set of all elements which are in both $\mathbf{A}$ and $\mathbf{B}$. It is written as $\mathbf{A} \cap \mathbf{B}$.

  $\mathbf{A} \cap \mathbf{B} = \{x \in \mathbf{U} \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\} = \{x \in \mathbf{A} \mid x \in \mathbf{B}\}$

  - For any set $\mathbf{A}$ the following hold:

    $$\mathbf{A} \cap \{\} = \{\} \text{ and } \mathbf{A} \cap \mathbf{U} = \mathbf{A}.$$

Operations on sets (2)

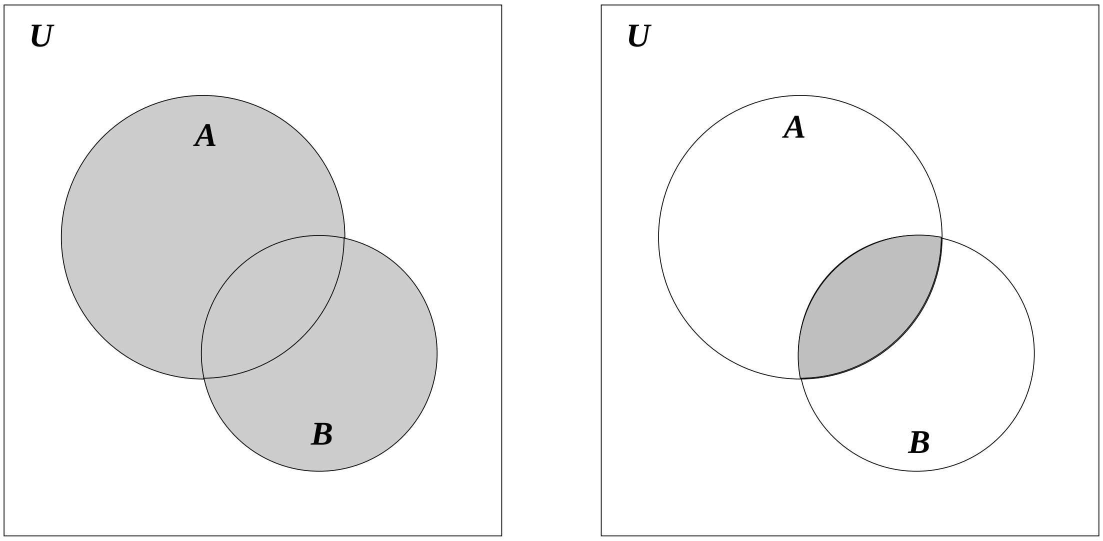

Figure 2: The union and intersection of sets $\mathbf{A}$ and $\mathbf{B}$

- *def:* If $\mathbf{A} \cap \mathbf{B} = \{\}$, then $\mathbf{A}$ and $\mathbf{B}$ are said to be *disjoint* sets.
- *def:* The *complement* of a set $\mathbf{A}$, denoted by $\overline{\mathbf{A}}$ (or $\sim \mathbf{A}$), is the set of all elements that are not in $\mathbf{A}$.
  $\overline{\mathbf{A}} = \{x \in \mathbf{U} \mid x \notin \mathbf{A}\}$

  Note that this definition of the complement of a set requires the existence of a Universal set.

Operations on sets (3)

- *def:* The *difference* of two sets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} - \mathbf{B}$ (or $\mathbf{A} \backslash \mathbf{B}$), is the set of elements which are in $\mathbf{A}$ but not in $\mathbf{B}$.
  $\mathbf{A} - \mathbf{B} = \{x \in \mathbf{U} \mid x \in \mathbf{A} \wedge x \notin \mathbf{B}\} = \{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

- *def:* The *boolean sum* of two sets $\mathbf{A}$ and $\mathbf{B}$ is the set $\mathbf{A} + \mathbf{B}$ consisting of all elements which are in $\mathbf{A}$ or in $\mathbf{B}$ but not in both.

  $\mathbf{A} + \mathbf{B} = \{x \in \mathbf{U} \mid (x \in \mathbf{A} \wedge x \notin \mathbf{B}) \vee (x \notin \mathbf{A} \wedge x \in \mathbf{B})\} =$
  $= (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$

Operations on sets (4)

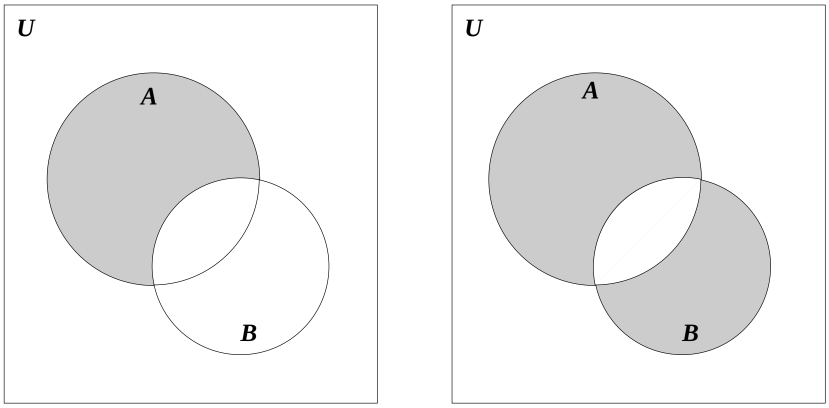

Figure 3: The difference and boolean sum of sets $\mathbf{A}$ and $\mathbf{B}$

Examples: Let $\mathbf{U}$ be $\{n \in \mathbf{Z} \mid 0 \leq n \leq 11\}$,
$\mathbf{A} = \{n \in \mathbf{U} \mid n < 7\}$, and $\mathbf{B} = \{n \in \mathbf{U} \mid 3 < n \leq 9\}$.

- $\overline{\mathbf{A}} = \{n \in \mathbf{U} \mid n \geq 7\}$ and $\overline{\mathbf{B}} = \{0, 1, 2, 3, 10, 11\}$
- $\mathbf{A} \cup \mathbf{B} = \{n \in \mathbf{U} \mid n \leq 9\}$
- $\mathbf{A} \cap \mathbf{B} = \{n \in \mathbf{U} \mid 3 < n < 7\}$
- $\mathbf{A} - \mathbf{B} = \{n \in \mathbf{U} \mid n \leq 3\}$ and $\mathbf{B} - \mathbf{A} = \{7, 8, 9\}$
- $\mathbf{A} + \mathbf{B} = \{0, 1, 2, 3, 7, 8, 9\}$

Laws of sets (1)

Let $\mathbf{U}$ be a universal set, and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be given sets.

- Idempotent laws
  $\mathbf{A} \cap \mathbf{A} = \mathbf{A}$
  $\mathbf{A} \cup \mathbf{A} = \mathbf{A}$

- Associative laws
  $(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C} = \mathbf{A} \cap (\mathbf{B} \cap \mathbf{C})$
  $(\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C} = \mathbf{A} \cup (\mathbf{B} \cup \mathbf{C})$

- Commutative laws
  $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$
  $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A}$

- Distributive laws
  $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$
  $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$

- Absorptive laws
  $\mathbf{A} \cap (\mathbf{A} \cup \mathbf{B}) = \mathbf{A}$
  $\mathbf{A} \cup (\mathbf{A} \cap \mathbf{B}) = \mathbf{A}$

Laws of sets (2)

- Identity laws
  $\mathbf{A} \cap \mathbf{U} = \mathbf{A}$ and $\mathbf{A} \cup \{\} = \mathbf{A}$
  $\mathbf{A} \cap \{\} = \{\}$ and $\mathbf{A} \cup \mathbf{U} = \mathbf{U}$
- Complement laws
  $\mathbf{A} \cap \overline{\mathbf{A}} = \{\}$ and $\mathbf{A} \cup \overline{\mathbf{A}} = \mathbf{U}$
  $\overline{\overline{\mathbf{A}}} = \mathbf{A}$, $\overline{\mathbf{U}} = \{\}$ and $\overline{\{\}} = \mathbf{U}$
- DeMorgan's laws
  $\overline{\mathbf{A} \cup \mathbf{B}} = \overline{\mathbf{A}} \cap \overline{\mathbf{B}}$.
  $\overline{\mathbf{A} \cap \mathbf{B}} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$.
- Alternative set difference representation
  $\mathbf{A} - \mathbf{B} = \mathbf{A} \cap \overline{\mathbf{B}}$
- Inclusion in union
  $\mathbf{A} \subseteq \mathbf{A} \cup \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A} \cup \mathbf{B}$
- Inclusion in intersection
  $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{A}$ and $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{B}$
- Transitive properties of subsets
  If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{C}$ then $\mathbf{A} \subseteq \mathbf{C}$